Midterm Exam Calculus 2

18 March 2021, 9:00-11:00

university of groningen

The midterm exam consists of 4 problems. You have 120 minutes to answer the questions. In addition you have 15 minutes to scan and upload your solutions to Nestor. Upload your solutions in a single file. For the filename, use the format Lastname_Studentnumber_Midterm. You can achieve 100 points which includes a bonus of 10 points.

1. [**5+5+10=20 Points**]

   Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

   $$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Let $\boldsymbol{u} = v\,\mathbf{i} + w\,\mathbf{j} \in \mathbb{R}^2$ be a unit vector, i.e. $v^2 + w^2 = 1$. Determine the directional derivative $D_{\boldsymbol{u}}f(0,0)$.

   (c) Use the definition of differentiability to determine whether f is differentiable at $(0,0)$.

2. [**10+5+10=25 Points**]

   Consider the curve parametrized by $\mathbf{r} : [0, \pi/2] \to \mathbb{R}^3$ with

   $$\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j} + (\cos^2 t - \sin^2 t)\,\mathbf{k}.$$

   (a) Determine the parametrization by arc length. You may use that $\frac{\mathrm{d}}{\mathrm{d}t}\sin^2 t = -\frac{\mathrm{d}}{\mathrm{d}t}\cos^2 t = 2\sin t\cos t$.

   (b) For each point on the curve, determine a unit tangent vector.

   (c) At each point on the curve, determine the curvature of the curve.

— please turn over —

3. **[5+10+10=25 Points]**

Let S be the ellipsoid in $\mathbb{R}^3$ defined by $x^2 + 2y^2 + 3z^2 = 6$.

(a) Compute the tangent plane of S at the point $(x_0, y_0, z_0) = (1, 1, 1)$.

(b) Use the Implicit Function Theorem to show that near the point $(x_0, y_0, z_0) = (1, 1, 1)$, the ellipsoid S can be considered to be the graph of a function f of x and y. Compute the partial derivatives of f with respect to x and y and show that the tangent plane found in (a) coincides with the graph of the linearization of f at $(x_0, y_0) = (1, 1)$.

(c) Use the method of Lagrange multipliers to determine the points on S where $g(x, y, z) = xy^2z^3$ has maxima and minima, respectively.

4. **[20 Points]**

Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4,\ y \geq 0\}$ and ∂D be the boundary of D oriented in the counterclockwise direction. For the vector field $\mathbf{F} : \mathbb{R}^2 \to R^2$, $(x, y) \mapsto P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ with $P(x, y) = 2y$ and $Q(x, y) = x$, verify

$$\int_{\partial D} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

by computing both sides of the equality. You may use that $\int \cos^2 t \,\mathrm{d}t = \frac{1}{2}(t + \sin t \cos t)$ and $\int \sin^2 t \,\mathrm{d}t = \frac{1}{2}(t - \sin t \cos t)$.

Solutions

1. (a) Using polar coordiantes $(x, y) = (r\cos\theta, r\sin\theta)$ we get for $r > 0$,

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r(\cos^3\theta - \sin^3\theta)$$

which goes to $0 = f(0,0)$ for $r \to 0$. Hence f is continuous at $(0,0)$.

(b) By definition

$$\begin{aligned} D_{\boldsymbol{u}}f(0,0) &= \lim_{t\to 0}\frac{f(tv,tw) - f(0,0)}{t} \\ &= \lim_{t\to 0}\frac{\frac{t^3v^3 - t^3w^3}{t^2v^2+t^2w^2} - 0}{t} \\ &= \lim_{t\to 0} v^3 - w^3 \\ &= v^3 - w^3. \end{aligned}$$

(c) Choosing $\boldsymbol{u} = (1,0)$ in part (b) we get $f_x(0,0) = 1$ and similarly choosing $\boldsymbol{u} = (0,1)$ we get $f_y(0,0) = -1$. The linearization of f at $(0,0)$ hence is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x - y.$$

For the differentiability of f at $(0,0)$ we need to study the limit of

$$\frac{f(x,y) - L(x,y)}{\|(x,y)\|}$$

for $(x,y) \to (0,0)$. For $(x,y) \neq (0,0)$ we have

$$\frac{f(x,y) - L(x,y)}{\|(x,y)\|} = \frac{\frac{x^3-y^3}{x^2+y^2} - (x-y)}{(x^2+y^2)^{1/2}} = \frac{x^3 - y^3 - (x-y)(x^2+y^2)}{(x^2+y^2)^{3/2}}.$$

Using polar coordinates we get for $r > 0$,

$$\begin{aligned} \frac{x^3 - y^3 - (x-y)(x^2+y^2)}{(x^2+y^2)^{1/2}} &= \frac{r^3\cos^3\theta - r^3\sin^3\theta - (r\cos\theta - r\sin\theta)(r^2\cos^2\theta + r^2\sin^2\theta)}{(r^2\cos^2\theta + r^2\sin^2\theta)^{3/2}} \\ &= \cos^3\theta - \sin^3\theta - (\cos\theta - \sin\theta) \end{aligned}$$

which for $\theta = \pi/4$ gives $\left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$ and for $\theta = -\pi/4$ gives $\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$ and hence has no limit for $r \to 0$. We conclude that f is not differentiable at $(0,0)$.

2. (a) The tangent vector

$$\begin{aligned} \mathbf{r}'(t) &= -3\cos^2 t\sin t\,\mathbf{i} + 3\sin^2 t\cos t\,\mathbf{j} + (-2\cos t\sin t - 2\sin t\cos t)\,\mathbf{k} \\ &= -3\cos^2 t\sin t\,\mathbf{i} + 3\sin^2 t\cos t\,\mathbf{j} - 4\cos t\sin t\,\mathbf{k} \end{aligned}$$

has length

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \left(9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 16\cos^2 t\sin^2 t\right)^{1/2} \\ &= \left(\cos^2 t\sin^2 t\left(9(\cos^2 t + \sin^2 t) + 16\right)^{1/2}\right. \\ &= \cos t\sin t\,(25)^{1/2} = 5\cos t\sin t, \end{aligned}$$

where we used that $\cos t$ and $\sin t$ are non-negative for $t \in [0, \pi/2]$. The arc length is hence

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\| \, \mathrm{d}\tau = 5 \int_0^t \cos\tau \sin\tau \, \mathrm{d}\tau = \frac{5}{2} \sin^2(\tau)\Big|_{\tau=0}^{\tau=t} = \frac{5}{2}\sin^2 t = \frac{5}{2}(1-\cos^2 t).$$

So $\sin^2 t = \frac{2}{5}s$ and $\cos^2 = 1 - \frac{2}{5}s$. The parametrization by arc length is thus given by

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = ((1 - \frac{2}{5}s)^{3/2}\,\mathbf{i} + (\frac{2}{5}s)^{3/2}\,\mathbf{j} + (1 - \frac{4}{5}s)\,\mathbf{k}$$

with $s \in [s(0), s(\pi/2)] = [0, 5/2]$.

(b) The unit tangent vector at the point $\tilde{\mathbf{r}}(s)$, $s \in [0, 5/2]$, is given by

$$\mathbf{T}(s) = \frac{\mathrm{d}\tilde{\mathbf{r}}(s)}{\mathrm{d}s} = -\frac{3}{5}\left(1 - \frac{2}{5}s\right)^{1/2}\mathbf{i} + \frac{3}{5}\left(\frac{2}{5}s\right)^{1/2}\mathbf{j} - \frac{4}{5}\mathbf{k}$$

which agrees with

$$\begin{aligned}\frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) &= \frac{1}{5\cos t \sin t}\left(-3\cos^2 t \sin t\,\mathbf{i} + 3\sin^2 t\cos t\,\mathbf{j} - 4\cos t\sin t\,\mathbf{k}\right) \\ &= -\frac{3}{5}\cos t\,\mathbf{i} + \frac{3}{5}\sin t\,\mathbf{j} - \frac{4}{5}\mathbf{k} \qquad (1)\end{aligned}$$

when substituting $\sin^2 t = \frac{2}{5}s$ and $\cos^2 = 1 - \frac{2}{5}s$.

(c) The curvature at the point $\tilde{\mathbf{r}}(s)$, $s \in [0, 5/2]$, is given by

$$\begin{aligned}\kappa(s) &= \left\|\frac{\mathrm{d}\mathbf{T}(s)}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{3}{5}\left(1 - \frac{2}{5}s\right)^{1/2}\mathbf{i} + \frac{3}{5}\left(\frac{2}{5}s\right)^{1/2}\mathbf{j} - \frac{4}{5}\mathbf{k}\right)\right\| \\ &= \left\|-\frac{3}{5}\left(-\frac{2}{5}\right)\frac{1}{2}\left(1 - \frac{2}{5}s\right)^{-1/2}\mathbf{i} + \frac{3}{5}\frac{2}{5}\frac{1}{2}\left(\frac{2}{5}s\right)^{-1/2}\mathbf{j} + 0\,\mathbf{k}\right\| \\ &= \frac{3}{25}\left((1 - \frac{2}{5}s)^{-1} + (\frac{2}{5}s)^{-1}\right)^{1/2} \\ &= \ldots \\ &= \frac{3}{5}\left(\frac{1}{10s - 4s^2}\right)^{1/2}.\end{aligned}$$

Differentiating (1) with respect to t and dividing by $\|\mathbf{r}'(t)\|$ which by the chain rule corresponds to differentiating with s gives

$$\frac{1}{5\cos t \sin t}\left(\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t\right)^{1/2} = \frac{3}{25\cos t \sin t}$$

which agrees with the κ above when substituting $\sin^2 t = \frac{2}{5}s$ and $\cos^2 = 1 - \frac{2}{5}s$.

3. (a) Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$. Then the ellipsoid S is given by the equation $F(x, y, z) = 6$. The gradient of F is $\nabla F(x, y, z) = 2x\,\mathbf{i} + 4y\,\mathbf{j} + 6z\,\mathbf{k}$ and is normal to S at $(x, y, z) \in S$. The tangent plane at $(x_0, y_0, z_0) \in S$ is given by the equation

$$\nabla F(x_0, y_0, z_0) \cdot ((x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}) = 0$$

i.e.

$$2x_0(x - x_0) + 4y_0(y - y_0) + 6z_0(z - z_0) = 0.$$

For $(x_0, y_0, z_0) = (1, 1, 1)$ this gives

$$x + 2y + 3z = 6. \tag{2}$$

(b) For F in part (a), we have

$$\frac{\partial F}{\partial z}(x_0, y_0, z_0) = 6z_0 = 6 \neq 0\,.$$

By the Implicit Function Theorem S is near $(x_0, y_0, z_0) = (1, 1, 1)$ locally the graph of a function $f : (x, y) \mapsto z = f(x, y)$. The implicit function f has partial derivatives at $(x_0, y_0) = (1, 1)$ given by

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\frac{\frac{\partial F}{\partial x}(x_0, y_0, z_0)}{\frac{\partial F}{\partial z}(x_0, y_0, z_0)} = -\frac{2x_0}{6z_0} = -\frac{1}{3}$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) = -\frac{\frac{\partial F}{\partial y}(x_0, y_0, z_0)}{\frac{\partial F}{\partial z}(x_0, y_0, z_0)} = -\frac{4y_0}{6z_0} = -\frac{2}{3}.$$

The linearization of f at $(x_0, y_0) = (1, 1)$ is given by

$$\begin{aligned} L(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\ &= 1 - \frac{1}{3}(x - 1) - \frac{2}{3}(y - 1) \\ &= 2 - \frac{1}{3}x - \frac{2}{3}y. \end{aligned}$$

The graph of L is given by the equation $L(x, y) = z$ which agrees with (2).

(c) Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$ and $g(x, y, z) = xy^2z^3$. A point (x, y, z) being a critical point of g restricted to the ellipsoid $F(x, y, z) = 6$ is equivalent to the existence of a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\lambda\nabla F(x, y, z) = \nabla g(x, y, z)$. Together with the constraint $F(x, y, z) = 6$ this gives the following four scalar equations:

$$\begin{aligned} \lambda F_x(x, y, z) &= g_x(x, y, z), \\ \lambda F_y(x, y, z) &= g_y(x, y, z), \\ \lambda F_z(x, y, z) &= g_z(x, y, z), \\ x^2 + 2y^2 + 3z^2 &= 6 \end{aligned}$$

i.e.

$$\begin{aligned} 2\lambda x &= y^2z^3, \\ 4\lambda y &= 2xyz^3, \\ 6\lambda z &= 3xy^2z^2, \\ x^2 + 2y^2 + 3z^2 &= 6. \end{aligned} \tag{3}$$

For $x = 0$, $y = 0$ or $z = 0$, we have $g(x, y, z) = 0$. On the other hand $g(1, 1, 1) = 1 > 0$ and $g(-1, 1, 1) = -1 < 0$. So the maxima and minima cannot have $x = 0$, $y = 0$ or $z = 0$ and we can exclude such points from the solutions of (3). Then the first three equations of (3) yield

$$\frac{y^2z^3}{x} = xz^3 = xy^2z.$$

Hence $y = x^2$ and $z^2 = y^2 = x^2$. Filling this into $F(x, y, z) = 6$ gives $x^2 + 2x^2 + 3x^2 = 6x^2 = 6$, i.e. $x^2 = 1$ and similarly $y^2 = z^2 = 1$. For $(x, y, z) = (1, \pm 1, 1)$ and $(x, y, z) = (-1, \pm 1, -1)$, $g(x, y, z) = 1$ and for $(x, y, z) = (-1, \pm 1, 1)$ and $(x, y, z) = (1, \pm 1, -1)$, $g(x, y, z) = -1$. By the Weierstrass Extreme Value Theorem these points are thus maxima and minima, respectively.

4. We first evaluate the left side of the equation. The boundary ∂D is piecewise smooth and consists of the smooth pieces parametrized by

$$\mathbf{r}(t) = (2\cos t, 2\sin t) =: (x(t), y(t)), \quad t \in [0, \pi]$$

and

$$\tilde{\mathbf{r}}(t) = (t, 0) =: (\tilde{x}(t), \tilde{y}(t)), \quad t \in [-2, 2],$$

respectively. Hence

$$\begin{aligned}
\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_0^\pi F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t + \int_{-2}^2 F(\tilde{\mathbf{r}}(t)) \cdot \tilde{\mathbf{r}}'(t)\,\mathrm{d}t \\
&= \int_0^\pi \left(P(\mathbf{r}(t))x'(t) + Q(\mathbf{r}(t))y'(t)\right)\,\mathrm{d}t \\
&\quad + \int_{-2}^2 \left(P(\tilde{\mathbf{r}}(t))\tilde{x}'(t) + Q(\tilde{\mathbf{r}}(t))\tilde{y}'(t)\right)\,\mathrm{d}t \\
&= \int_0^\pi \left(2 \cdot 2\sin t \cdot (-2\sin t) + 2\cos t\, 2\cos t\right)\,\mathrm{d}t \\
&\quad + \int_{-2}^2 (0 \cdot 1 + t \cdot 0)\,\mathrm{d}t \\
&= \int_0^\pi \left(-8\sin^2 t + 4\cos^2 t\right)\,\mathrm{d}t \\
&= \int_0^\pi \left(-8 + 8\cos^2 t + 4\cos^2 t\right)\,\mathrm{d}t \\
&= \int_0^\pi \left(12\cos^2 t - 8\right)\,\mathrm{d}t \\
&= \frac{12}{2}(\cos t \sin t + t) - 8t \Big|_{t=0}^{t=\pi} \\
&= -2t\big|_{t=0}^{t=\pi} \\
&= -2\pi.
\end{aligned}$$

For the right side of the equality we have

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,\mathrm{d}A &= \iint_D (1 - 2)\,\mathrm{d}A \\
&= -\text{area of } D \\
&= -\frac{1}{2}\pi 2^2 \\
&= -2\pi
\end{aligned}$$

which agrees with the left side of the equality.